Gödel's Theorem:

An Incomplete Guide to Its Use and Abuse

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This unique exposition of Kurt Gödel's stunning incompleteness theorems for a general audience manages to do what no other has accomplished: explain clearly and thoroughly just what the theorems really say and imply and correct their diverse misapplications to philosophy, psychology, physics, theology, post-modernist criticism and what have you.

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Introduction

- Gödel’s incompleteness theorem is probably the one that has arose more interest among non-mathematicians, it was published in 1931 by Gödel, and strengthened in 1936 by Rosser

- That’s a result of logic about the *consistency* and *completeness* of *formal systems*, those terms have a precise technical meaning and so care is needed when interpreting the theorem outside logic

- The implications of this theorem are often overstated, for example: “turned not only mathematics, but also the whole world of science on its head”

- Gödel’s incompleteness theorem brought no revolution whatsoever to mathematics, in general it plays no role in the work of working mathematicians

- However, the theorem raises a number of philosophical questions concerning the nature of logic and mathematics

The First Incompleteness Theorem

Terminology

- Given a formal language L, a *formal system S in L* is a set of *axioms* and syntactical *rules of reasoning* (also called *inference rules*) expressed in L.
- A *theorem of S* is any statement T of S which is obtainable by a series of applications of the inference rules of S starting from the axioms of S, such a sequence is called a *proof of T in S*
- Those concepts are *purely symbolic*, nothing is said about any meaning or soundness

- A formal system S is said to be *consistent* if it has no sentence A such that A and ¬A are both theorems of S

- A sentence A of S is said to be *undecidable in S* if neither A nor ¬A are theorems of S

- A formal system is said to be *complete* if it has no undecidable sentences, otherwise it is *incomplete*

- If A is a sentence of S, we denote by S+A the formal system obtained by adding A as a new axiom to the ones of S, and with the same rules

Observations

- By *ex falso quodlibet* an inconsistent theory has no undecidable statements, everything is a theorem
- A is probable in S iff S+¬A is inconsistent
- A is undecidable in S iff both S+A and S+¬A are consistent

First Incompleteness Theorem

- **First incompleteness theorem** (**Gödel-Rosser**)**.** Any consistent formal system S within which a certain amount of elementary arithmetic can be carried out is incomplete with regard to statements of elementary arithmetic

Goldbach-like Statements

- Goldbach's conjecture states that every even number greater than 2 is the sum of two primes, it is unknown whether it holds
- For any given even number greater than 2 we can check that property applying an algorithm, that's a *computable* property
- We call *Goldbach-like* those statements of the form "Every natural number has property P", where P is a computable property

- Goldbach-like statements have an important characteristic, if they are false they are disprovable because there’s an algorithm that finds a counterexample

- However, we do not know in advance which mathematical methods will give a proof of a true Goldbach-like statement, in case there is one

The Twin Prime Conjecture

- The Twin Prime conjecture states there are infinitely many primes p such that $p+2$ is also prime, it is unknown whether it holds
- That is not a Goldbach-like statement: given a natural n the procedure that systematically looks for pairs p, $p+2$ with $n \leq p$ will not terminate if there's none

An Important Property

- If a Goldbach-like sentence in a consistent system S incorporating some basic arithmetic is provable or undecidable in S then it is necessarily true, for if it was false a proof of its negation would exist, and so S wouldn’t be consistent
- In the traditional proof of the incompleteness theorem a Goldbach-like and undecidable sentence is shown

Hilbert's *non ignorabimus*

- Gödel's Incompleteness theorem does not refute Hilbert's view that mathematicians can find solutions to any mathematical problem by means of pure reason
- Instead, it establishes that Hilbert's optimism cannot be justified by exhibiting any single formal system within which all mathematical problems are solvable

Unprovable Truths

- It is often said that Gödel demonstrated that there a truths that cannot be proved. This is incorrect, “provability” is always relative to a formal system
- If an arithmetic sentence was undecidable in ZFC we still could select a stronger set of axioms valid as foundations and try to prove it there
- That is not to say that such a switch would be without controversy

Complete Formal Systems

- The incompleteness theorem does not imply that *every* consistent formal system is incomplete
- The theory of the real numbers is complete, and it comprises the arithmetic of real numbers
- Although the natural numbers are a subset of the real numbers they are not *definable* within the theory of real numbers, and so the premise of the incompleteness theorem do not hold

Applicability

- The incompleteness theorem guarantees the existence of undecidable *arithmetical* statements in certain kinds of formal systems
- It says nothing about the existence of undecidable non-arithmetical or non-mathematical statements
- The incompleteness theorem does not apply in contexts where there's no formal system like, say, the Bible, or the Constitution, analogies and metaphors rarely suffice

Mind vs. Computer

- Some people speculate that the incompleteness theorem implies the mind surpasses the machine because it can see the truth of formulas the machine won't be able to demonstrate
- This argument is invalid, in general we have no idea about whether the Gödel sentence of a system is true, what we know is that it is true iff the system is consistent, and this much is provable in the system itself

Incompleteness and the TOE

- Some people question the feasibility of a Theory of Everything in physics because of the theorem
- A TOE would probably be a formal system where Gödel’s theorem applies, and in such case the theorem tells us there would be an incompleteness in its arithmetical component, but whether or not the basic equations of physics are complete considered as a description of the physical world, and what completeness might mean in such a case, is not something that the incompleteness theorem tells us anything about

Truth

- “Truth” is not a mathematical concept, and so it is normally avoided by mathematicians, a proof in logic is a syntactic derivation and has nothing to do with the “truth” or “falsehood” of a sentence
- When a mathematician says that the twin prime conjecture may be “true” he means that there may be infinitely many primes p such that $p+2$ is also prime, no more no less

ZFC

- There's nothing that supports in that arithmetical statements like the twin prime conjecture could be actually undecidable in ZFC, in practice there's no need to feel at all worried by the possibility of natural mathematical problem being unsolvable
- Some problems in Set Theory are known to be unsolvable in ZFC, like Cantor's continuum hypothesis

Some Later Developments

- There has been considerable work seeking undecidable statements close to “ordinary mathematics”
- In arithmetic the first result of this kind was the *Paris-Harrington theorem* (1976) which establishes the unprovability of a combinatorial statement
- Gödel himself suggested a way to extend the arithmetical component of ZFC with *axioms of infinity* that imply some undecidable statements

- Another line of development—followed by Gregory Chaitin in the 1960s—relates incompleteness to Kolmogorov complexity

The Second Incompleteness Theorem

Second Incompleteness Theorem

- **Second incompleteness theorem (Gödel).** For any consistent formal system S within which a certain amount of elementary arithmetic can be carried out, the consistency of S cannot be proved in S itself

Observations

- The “certain amount of arithmetic” in the second theorem is not the same we ask for in the first one
- “S is consistent” can be expressed in said systems thanks to a technique called *arithmetization of syntax* that uses a way of representing syntactical objects such as sentences and proofs as numbers called *Gödel numbering*

- The consistency of such systems is sometimes proved in other systems, for instance Gentzen showed that PA is consistent using a theory T that extends PA, but that yields an infinite regress since we need now to investigate the consistency of T

Doubts

- Some people think the second incompleteness theorem raises doubts about the consistency of formal systems used in mathematics, we can't prove it in an absolute manner, so there's a logical possibility of mathematics being inconsistent
- From a metamathematical view, though, it would be of little use to prove the consistency of ZFC within ZFC, for if we doubt about its consistency in the first place how can we trust the theorem?

- In fact, if ZFC was inconsistent it would indeed proof its consistency, since every sentence is a theorem in an inconsistent system, so even if ZFC could prove its own consistency little would follow as far as our confidence in the system go

- The second incompleteness is mainly of interest to logicians, whose object of study are logics, the same way an algebraist studies, say, rings and is interested in their properties

So, Are Mathematics At Risk?

- No one doubts of the consistency of the axioms on which ordinary mathematics are based, and therefore of the validity of the theorems such as that there are infinitely many primes
- That belief comes from consensus, common sense, experience, tradition, etc., not from logic

Hilbert’s Program

- It is often said that the second incompleteness theorem demolished Hilbert’s program, whose goal was to prove the consistency of mathematics by *finitistic* reasoning
- This was not the view of Gödel himself. Rather, the theorem showed that the means by which acceptable consistency proofs could be carried out had to be extended and Gödel gave one way in his “Dialectica interpretation,” published in 1958

Computability, Formal Systems and Incompleteness

Alternative Proofs

- Over time logicians have developed techniques to demonstrate both incompleteness theorems using other approaches than the original proofs

Strings

- By a *string* is meant any finite sequence of symbols
- The *length* of a string is defined to be the number of occurrences of its symbols
- For instance numerals in decimal positional notation are strings of the alphabet of digits

Computably Enumerable Sets

- Numerals can be generated in lexicographic order by a computer program, such a set of strings is called *computably enumerable*
- In that definition we allow repetitions, and if the set is finite we allow the program not to halt, as long as it eventually exhausts the set of strings
- Turing machines or some other model of computation are used to formalize these concepts, but we informal descriptions will do for us

Computably Decidable Sets

- There is an algorithm that decides whether a given string will *ever* appear in the enumeration of numerals mentioned in the previous slide—“13” does, “007” does not, neither does “foo”—by definition this property makes the set of numerals *computable*, or *computably decidable*, or just *decidable*
- A set that is not decidable is said to be (*computably*) *undecidable*

We’ve Got Two “Decidable”s

- We have by now a definition of *decidable* that applies to sentences of formal systems, an another one that applies to sets of strings
- They are different, but there’s a connection between them that we will see in this section

Enumerability & Decidability

- Every computably decidable set is computably enumerable
- A set E is computably decidable iff both E and its complement are computably enumerable
- **Undecidability theorem (Turing, Church).** There are computably enumerable sets which are not computably decidable

Formal Systems and Decidability

- The set of sentences of the language of a formal system is assumed to be a decidable set of strings
- Axioms and inference rules need to be defined in such a way that the set of theorems is computably enumerable
- If the set of theorems is indeed decidable the formal system is said to be *decidable*

- A complete formal system is always decidable, for either it is inconsistent, or else we determine if a sentence A is a theorem enumerating its theorems until either A or $\neg$A is found

- There are also theories which are decidable and incomplete

Computability & Incompleteness

- There is a proof of the incompleteness theorem that does not use any arithmetical formalization of self-referential sentences (so they are not essential)
- It is based on computability theory and uses the Matiyasevich-Robinson-Davis-Putman theorem
- The proof shows that there are infinitely many equations $D(x_1, ..., x_n) = 0$ for which is undecidable in S whether or not they have solution

The Completeness Theorem

The Completeness Theorem

- Gödel himself proved that first-order theories are complete, in that theorem "complete" has another meaning though
- A *model* of a first-order system is a mathematical structure that satisfies its axioms given a certain interpretation
- Given a first-order system S, the completeness theorem establishes that a sentence of S is a theorem of S iff it holds in all models of S

Nonstandard Models

- Sometimes it is said that the first incompleteness theorem implies that in any theory T of a certain degree of complexity there are sentences that are true in all models of T yet undecidable in T
- The first incompleteness theorem neither states nor implies such a thing, on the contrary, the completeness theorem implies that if a sentence is true in all models of a first-order theory T (such as PA or ZFC) then it is decidable in T, for it is indeed a theorem of T